# Technical note 2

# Analysis of a signal in the frequency domain.
# The concept of Fourier series expansion and transform.
# Applications to the analysis of a single channel surface EMG signal.

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**This technical note is aimed to readers with a high-school knowledge of mathematics and physics of electrical phenomena and signals.**
**For this reason, terminology and some concepts are oversimplified and their mathematical explanation is not provided.**

**Outline**
1. Concept and properties of the sinusoid.
2. Sampling a signal in time.  Aliasing.
3. Expansion of a periodic signal into e series of sinusoids. Amplitude and power spectra.
4. Expansion of a random signal into a series of sinusoids. Amplitude and power spectra.
5. The effect of sampling frequency and duration of a signal on its Fourier transform.
6. Applications examples. Applications to the sEMG signal.
7. The cross-spectrum of two signals.

## 1. Concept and properties of the sinusoid.

Consider the vector **A** in Fig. 1. This vector has a "magnitude" A and a direction or angle, or "phase", φ.   This vector also has two components in the x-y plane: they are $A_x$=A cos φ along x and $A_y$=A sin φ along y.  The "magnitude", or "modulus", of the vector is given by Pythagoras' Theorem as $A = \sqrt{A_x^2 + A_y^2}$ while its phase is the angle whose tangent is sin φ/cos φ , that is φ= arctg($A_y$/$A_x$).
Furthermore consider that an angle can be measured in either degrees or radians where a degree is a full-turn/360 and a radian is the angle cutting an arc equal to the circumference's radius. Since a circumference contains 2π radii it follows that 1 rad = 360° / 2π = 57.3° and the following proportion holds: 2π : 360° = rad : deg  where rad is an angle expressed in radians and deg is the same angle expressed in degrees. Given either measure the other can be computed.  In signal processing angles are usually measured in radians.

Consider now that the vector A is rotating counterclockwise so that its magnitude A remains the same but φ increases proportionally to time so that it takes T seconds to complete a full turn, that is $360^o$ or 2π radians. That means that the vector will make f full revolutions per second and therefore f=1/T and T=1/f. The quantity f is the frequency of rotation and is measured in cycles/s or Hertz (Hz). For example. If T =0.1 s it will be f= 10 Hz and if f=1000Hz then T= 1ms. The quantity T, expressed in seconds, is called the "period" of the sinusoid. It follows that the angular velocity of φ is ω = 2πf = 2π/T where ω is the angular velocity or "angular frequency", expressed in radians/s (instead of cycles/s) .
For example, if A makes 3 full revolutions per second il will be f=3Hz and ω=6π radians/s and the period will be 1/3 of a second. The "velocity" of φ, expressed in rad/s is ω and therefore it will be φ = ω t = 2πtf= 2πt/T. The following proportions hold : φ : $360^o$ = t: T if φ is expressed in degrees, and φ : 2π = t: T , if φ is expressed in radians, as more usual.

Consider now how the two components of the vector A evolve as functions of the angle φ, that is as functions of time since φ = ω t, where ω = 2πf.
The plot of $A_y$ = A sin φ = A sin 2πf t = A sin 2πt/T is a "sinusoid" or "sine wave" and is depicted in green in Fig. 1.
The plot of $A_x$ = A cos φ = A cos 2πf t = A cos 2πt/T is a "cosinusoid" or "cosine wave" and is depicted in red in Fig. 1.
Observe that a cosinusoid is a sinusoid shifted left by a quarter of a period. Also observe that the values of a sinusoid or of a cosinusoid are comprised between -A and +A since sin φ and cos φ are comprised between -1 and +1.

The sinusoid is a very important mathematical function because of its many properties. For example, the derivative (that is the function describing the slope or rate of change) of a sinusoid is a cosinusoid and viceversa, with a negative sign.

The electrical power that would be transformed in heat by a sinusoidal voltage applied to a 1 Ω resistor is called the "power" of the sinusoid and is given by $A^2/2$. The square root of this value is called the root mean square value of the sinusoid (RMS) and is A/√2= 0.707A. Note that the RMS value is related to the amplitude only and not to the frequency of the sinusoid. The concept of RMS applies to any waveform but the RMS = 0.707A applies to sinusoids only.

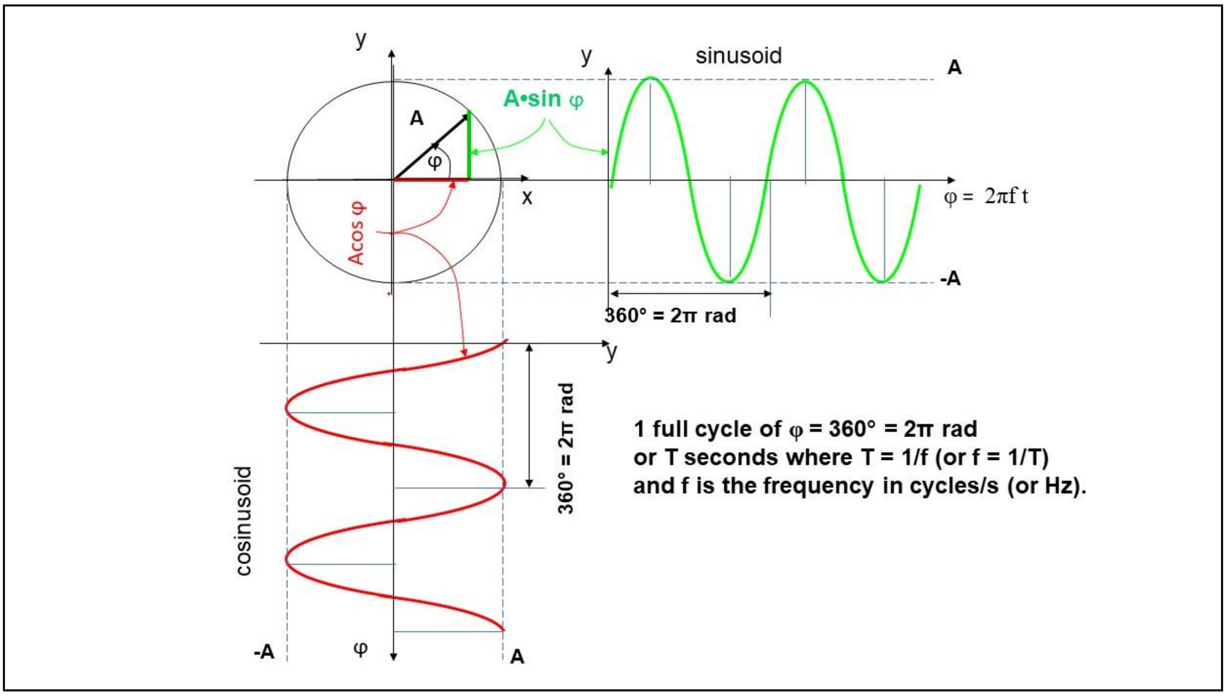


Fig. 1. Generation of a sinusoid and a cosinusoid as components along the x and y axis ($A_x$ and $A_y$) of a rotating vector **A**.

## 2. Sampling a signal in time. Aliasing.

Consider the signal y = g(t) depicted with a dashed black line in Fig. 2a. This signal is continuous in time and cannot be processed by a computer. Computers can only process numbers so this signal must be translated into a sequence of numbers. In order to do this the signal is “sampled” and a sequence of “samples” equally spaced in time is generated. This sequence is represented by the vertical lines in Fig 2a. Each sample has a numerical value (the length of each vertical bar) which is translated into a binary number by an analog-to-digital converter (ADC). This number can be processed by a computer. Of course, we miss information about what the signal is doing in between samples. However, if the samples are “sufficiently close” we can reconstruct the missing information by “linear interpolation”, that is by connecting the sampled values by segments or arcs. In the case of Fig. 2a this procedure produces a new signal that is “sufficiently close” to the original signal because the straight lines (not indicated in Fig. 2a) substantially overlap with the original signal with a negligible error. This is not the case if the interval between samples is twice that used in Fig. 2a, as shown in Fig. 2b where the sample frequency is half (one every other sample is taken).

The number of samples taken every second is the sampling frequency $f_s$ (or "sampling rate" measured in samples/s or, less correctly, in Hz). The time interval between adjacent samples is the sampling interval $t_s$ (measured in seconds), and is $t_s = 1/f_s$. In Fig. 2a the sequence of samples provides an "acceptable representation" of the original signal and further processing can be performed correctly. For example, if $f_{s1}$ =2000 Hz, $t_{s1}$ = 1/2000 = 0.5 ms.

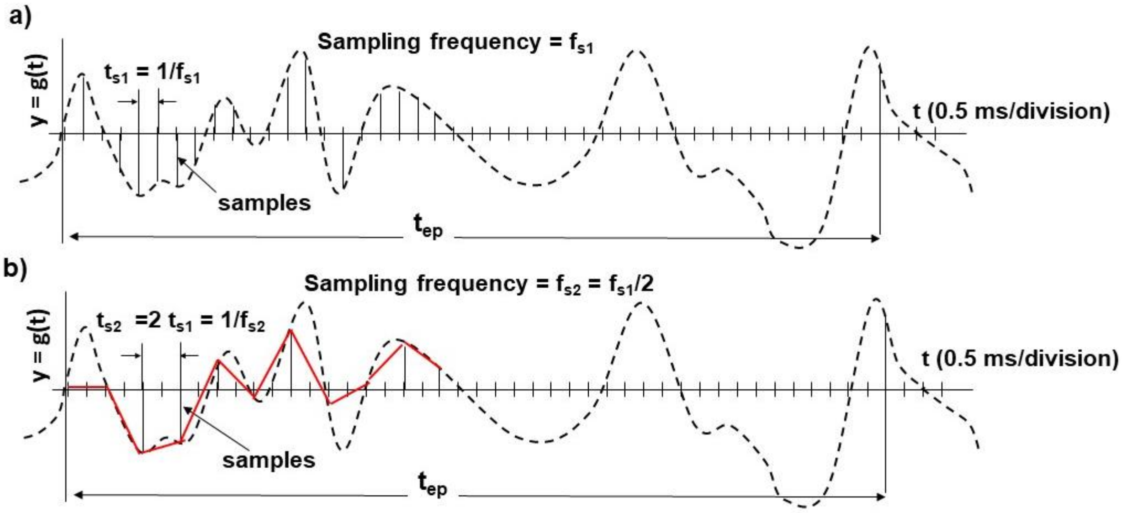


Fig. 2. Sampling a signal in time. a) the continuous signal represented by the dashed black line is sampled at the frequency $f_{s1}$ (for example 2000 Hz) with a sampling interval $t_{s1}$ (for example 0.5 ms) for an "epoch" of duration $t_{ep}$. We can see that the missing information in between samples does not cause a great error because the samples capture the features of the signal and its waveshape can be reconstructed faithfully by connecting the samples with straight lines or arcs (interpolation).
b) the same signal depicted in a) is sampled at half the sampling frequency used in a). We can see that the missing information in between samples causes a great error because the samples are too far apart to capture the features of the signal and its waveshape cannot be reconstructed faithfully by connecting the samples with straight lines (red line).

Fig. 2b depicts the same signal (dashed black line) sampled at half the frequency. For example, $f_{s2} = f_{s1} /2$ = 1000 Hz and $t_{s2} = 2\ t_{s1}$ = 1/1000 = 1 ms. The red lines connecting the sample values indicate that the sampled signal does not represent well the original signal and the "sampling" error is quite large. Processing this sampled signal would lead to incorrect results because of incorrect choice of the sampling frequency.

So, how do we know how to properly choose the sampling frequency of a continuous signal ? The answer to this important question will be provided in section 5.

Since we do not know what the signal is doing in between two samples it is possible that the sequence of samples might represent two or more signals having the same samples but different behavior in between samples. This is shown in Fig 3 where the sequence of samples (vertical black lines) applies to both sinusoid 1 (blue) and sinusoid 2 (red). Sinusoid 2 is called an "alias" of sinusoid 1 and this phenomenon is called "aliasing".

We are taking 8 samples in 50 ms so the sampling frequency is 160 Hz and the sampling interval is 6.25 ms. This is a proper choice of sample frequency for sinusoid 1 whose frequency is 20 Hz but a wrong choice for sinusoid 2 whose frequency is 180 Hz. Note that sinusoid 1 is sampled properly and can be reasonably well reconstructed (interpolated) by curve 3 (connecting the samples) while sinusoid 2 is not at all recovered by interpolation between samples. In other words, sinusoid 2, which has a frequency of 180 Hz, after being improperly sampled, as in the example indicated in Fig. 3, appears as a sinusoid having frequency of 20 Hz.

When we have only the sequence of samples, how do we know if it is coming from sinusoid 1 and not from its alias sinusoid 2? The answer is that we cannot distinguish the two cases. Therefore, if we are interested in sinusoid 1 we must be sure that sinusoid 2 is not present, while if we are interested in sinusoid 2 (and 1) we must substantially increase the sampling frequency. In case we were interested in sinusoid 2 (in absence of sinusoid 1) this sampling would give us curve 3 which, in this case, would be totally incorrect. This issue will be further discussed in section 5.

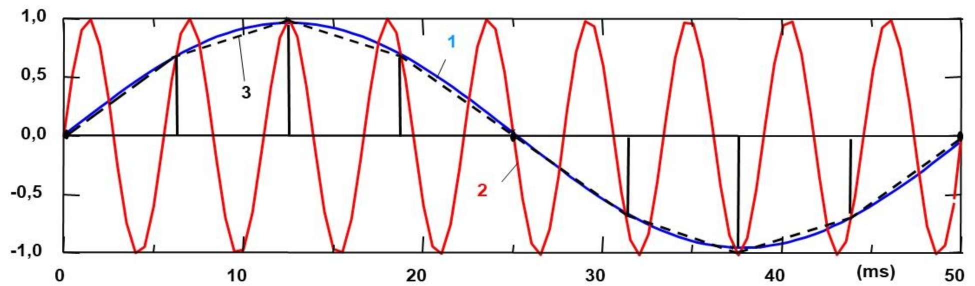


Fig. 3. Sampling frequency must be chosen after knowing the frequency of the signals we wish to sample. The sequence of samples indicated by the vertical black bars could be generated either by sampling sinusoid 1 or sinusoid 2. In BOTH cases the resulting interpolated signal would be curve 3 which is a good representation of sinusoid 1 but a totally incorrect representation of sinusoid 2.

## 3. Expansion of a periodic signal into e series of sinusoids. Amplitude and power spectra.

A periodic signal is a signal that repeats identically every T seconds, with a frequency f =1/T cycles/s or Hz, such as the one indicated in red in Fig. 3a where two cycles, or periods, of this signal s(t) are depicted. The signal has a period of 0.1 s and therefore a frequency of 10 Hz or 10 cycles/s. A sinusoid or a cosinusoid are special types of periodic signal.

A fundamental property of any periodic signal is that it can be "decomposed" or (and better) "expanded" into the sum of a number of sinusoids of proper amplitude and phase whose sum produces the original signal. This expansion of a signal into a sum of sinusoids is called the Fourier series expansion and the sinusoids are called "harmonics" of the signal. As indicated in Fig. 4a, the first harmonic has the same frequency f of the signal, the second has frequency double ( frequency 2f and period T/2), the third has frequency 3f and period T/3 , and so on. As a consequence, the spacing between adjacent harmonics is f. The number of harmonics necessary to reconstruct the original signal may range from very few to thousands, depending on the shape (or "waveform") of the signal. These harmonics are mathematical building blocks and *do not* mean that the device or organ producing the signal actually produced them and added them up to create the signal.

Consider now Fig. 4a, look at it from the right and draw a green vertical bar indicating the amplitude of each harmonic at the corresponding frequency (A1 at 10 Hz, A2 at 20Hz, and so on). We obtain a plot of amplitude of the harmonics versus their frequency. This plot is called the "amplitude spectrum" of the signal or "magnitude of the Fourier expansion" of the signal. An example is given in Fig. 4b for another signal having 8 harmonics (and there are no further harmonics). Each vertical bar represents the amplitude of one harmonic.

A "phase spectrum" (or "phase angle of the Fourier expansion of the signal") is obtained in the same way but is not reported in Fig.4. The amplitude and phase spectra are the "direct Fourier expansion" of the signal. They provide the representation of the signal "in the frequency domain" and contain the same information contained in the original signal (red curve in Fig.4a). They allow its perfect reconstruction. The reconstruction of the signal starting from its amplitude and phase spectra is called the "inverse Fourier transform" of the spectrum. It is important to understand and underline that these harmonics (or "spectral lines") are mathematical entities and, in general, do not represent (and are not generated by) any physiological mechanism or phenomena. This process can be applied to any signal (voice, music, earthquake waves, sounds produced by whales or ship propellers, activity of sunspots, blood pressure wave, EEG, ECG, EMG, etc). Although the harmonics are NOT individually generated by the physical or physiological source of the signal they are extremely useful in the process of "understanding" the signal and "processing" it to extract physical or physiological information. as shown in section 6.

Note that. if the original signal is sampled, its harmonics will be sampled as well at the same sapling frequency. Also note that an amplitude spectrum (Fig. 4b) could be defined by the dashed green line and the harmonics could be considered as "samples" of the spectrum.

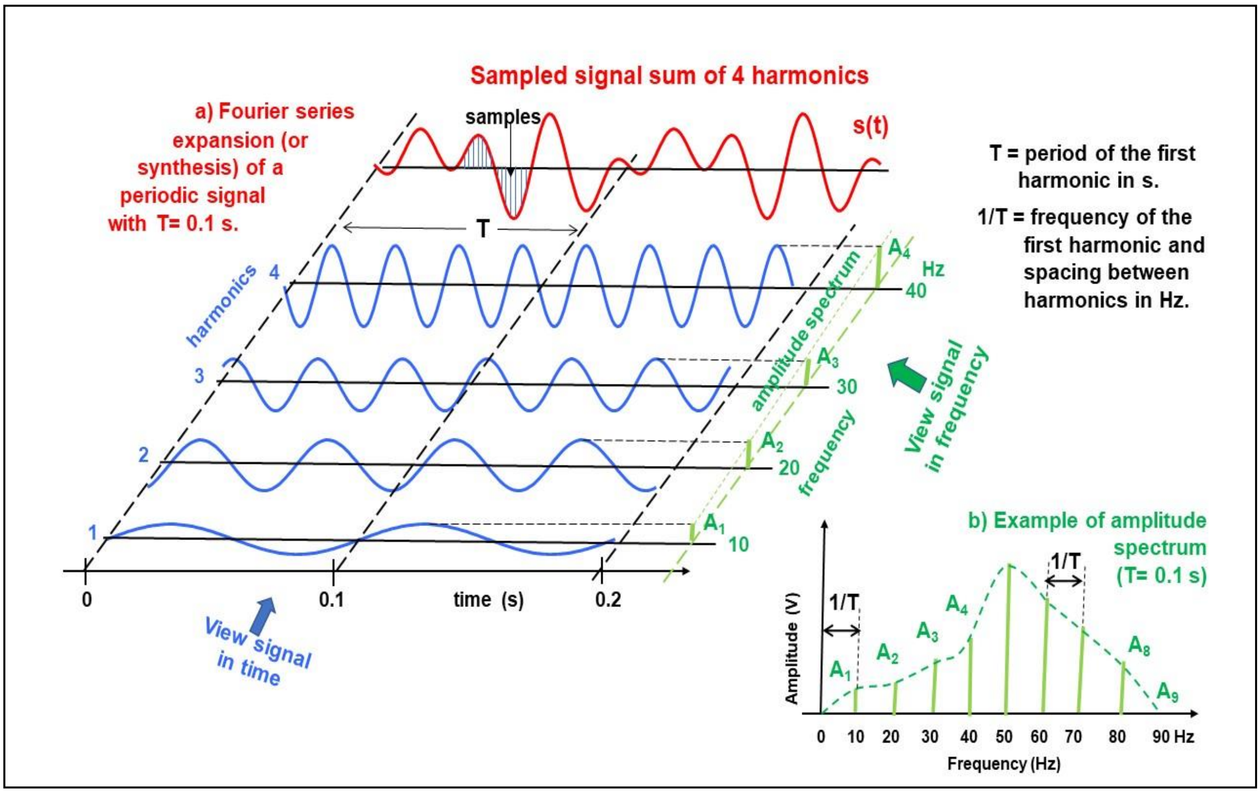


Fig.4. Fourier series expansion. A sampled periodic signal with period of T seconds (0.1s in the example, red line) can be decomposed into the sum of sinusoids called harmonics (4 in panel a)) whose frequencies are multiple of 1/T (10Hz in this example) as described in a). The amplitude spectrum is obtained as indicated in green in a). b) Amplitude spectrum of another signal having 9 harmonics. Note that these harmonics are mathematical entities and, in general, do not represent (and are not generated by) any physiological mechanism, organ or phenomena. See text for further explanation.

Note that if the period T of the signal is longer, with no change of signal waveshape, the harmonics will be closer to each other (because they are spaced by 1/T Hz) and the spectrum will be narrower. If the signal is irregular and with fast changes, the harmonics of higher frequency will have greater amplitudes than those of lower frequency.

The power of a sinusoid having peak value A was defined in section 1 as $A^2/2$. The same concept applies to the harmonics of a generic signal. The power of a signal is the sum of the powers of the individual harmonics. However, the RMS of a signal is **not** the sum of the RMS values of its individual harmonics because the square root of a sum is **not** the sum of the square roots of the addenda. If the original signal is sampled, all its harmonics are sampled at the same time instants. Fig. 5 shows the amplitude and powers spectra of two sinusoids having RMS values $V1_{RMS}$ and $V2_{RMS}$ respectively of 10 and 5 V and frequencies respectively of 25 Hz and 50 Hz. Note the amplitude spectrum showing two lines of amplitude 10 V and 5 V at 25 Hz and 50 Hz. Note the power spectrum showing two lines of 100 $V^2$ and 25 $V^2$ at the same frequencies. If taken simultaneously, the two pairs of lines represent the amplitude and power spectra of the sum of the two sinusoids.
Since the time and the frequency representations of a signal are perfectly equivalent, and one can be obtained from the other, what are then the reasons for a second representation? This question will be answered in section 5.

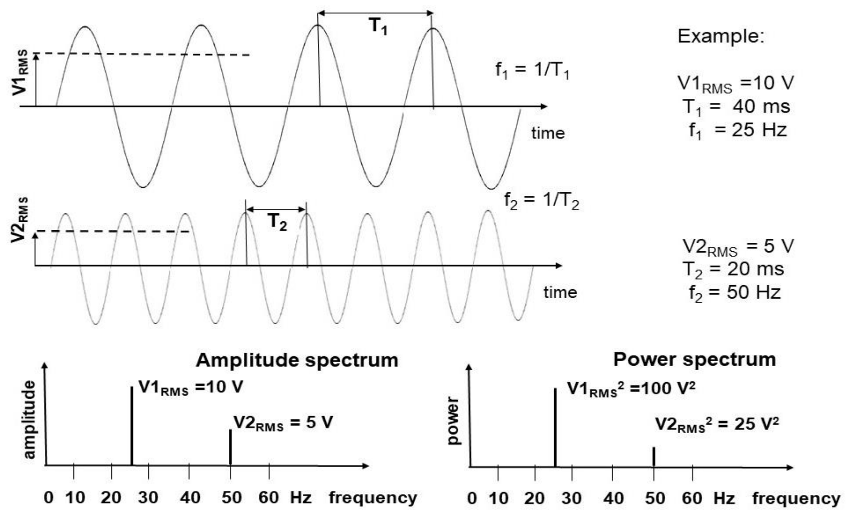


Fig. 5. Example of two sinusoids depicted versus time and in the "frequency domain". The amplitude and power spectra are shown. The sinusoids have the same starting phase. The phase spectrum (not shown) shows two values of zero degrees at 25 Hz and 50 Hz. Note that sinusoid V2 has amplitude that is ½ of that of V1 and power that is ¼ of that of V1.

## 4. Expansion of a random signal into a series of sinusoids. Amplitude and power spectra.

Most bioelectric signal (including sEMG and EEG) are not periodic and appear to be random (stochastic), such as Signa l and Signal 2 in Fig. 6. However, their analysis in the "frequency domain" (or frequency analysis" or "spectral analysis") can be performed with a mathematical "trick" consisting in taking a segment of the signal called "epoch" or "time window" of duration T seconds and assuming that it repeats periodically with period T. Although this is not true, this "hypothesis" allows a correct definition of the harmonics and of the signal amplitude and power spectra as outlined in Section 3 for a periodic signal.
Fig. 6 shows an example concerning the frequency analysis of two different random signals,

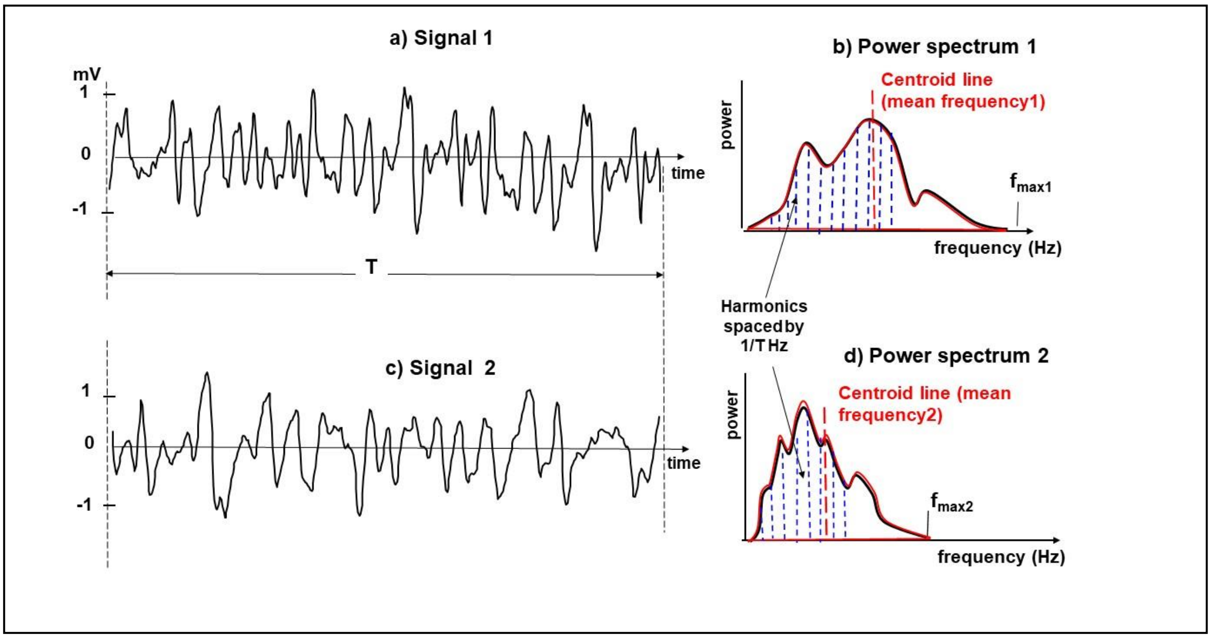


Fig. 6. Example of two random signals having similar amplitude but different power spectra. The abscissa of the centroid of the power spectrum (mean frequency) is an indicator of the spectral bandwidth of the signal. a) one epoch of duration T of Signal 1, b) one estimate of the power spectrum of Signal 1 with indication of the harmonics spaced by 1/T Hz. c) One epoch of duration T of Signal 2, d) one estimate of the power spectrum of Signal 2 with indication of the harmonics spaced by 1/T Hz. The harmonics of the two spectra have the same frequencies but different amplitudes. The frequency $f_{max}$ is the frequency of the highest harmonic of each spectrum.

Signal 1 and Signal 2. These two signals are sampled synchronously (samples not indicated for clarity), have similar amplitude (and therefore similar RMS values), but are obviously different. Signal 2 is "slower" and Signal 1 is "faster". How can we quantify this difference?

The answer comes from the power spectra plots of the two signals (right panels), which are quite different: one is narrower and the other is wider. The harmonics, indicated by the blue dotted vertical lines and separated by 1/T Hz in both cases, extend to different frequencies $f_{max1}$ and $f_{max2}$ and more power of Signal 1 is associated to higher frequency harmonics than Signal 2. This observation can be quantified in many ways. The most common is by means of the value of the centroid line (center of gravity or barycenter) of the power spectrum of each signal. This is called *mean frequency*, it is a weighed mean, and should not be misunderstood as the mean of the frequencies from 0 to $f_{max}$ (which is $f_{max}/2$). If we cut out the power spectrum of a signal on a piece of cardboard (red shapes in panels b) and d) ) and we balance it on a blade indicated by the red dashed line, the piece of cardboard will be in equilibrium. This single frequency value (centroid frequency or barycentre frequency) will give us some information about the width of the spectrum. This is not the only indicator used and other features provide additional quantitative information about the difference between two signals and their spectra. More detailed explanation and examples of application are provided in slides 41 to 52 in www.robertomerletti.it/en/emg/material/teaching/module7.

Spectra and their mean frequencies (or other features) are similar but not identical when computed from subsequent epochs of a signal. Since the signal is random, also its features have random values when their estimation is repeated on subsequent epochs. They are different, in subsequent epochs, even if the signal is "stationary". For this reason they are called "estimates" and fluctuate around a mean value that is called the "expected" value. This is a critical point when features are estimated from a "non stationary" signal that is changing the expected value of its features in time, such as the sEMG during dynamic contractions. This fact requires a careful choice of the epoch duration. Epoch duration (T) must be long enough to have a reasonable frequency resolution (1/T Hz) and be short enough so that the signal is reasonably stationary and is not significantly changing its RMS and spectrum during each of the subsequent epochs. In the case of sEMG, proper choice of epoch duration is a compromise that depends on the rate of force change or velocity of shortening or lengthening of the muscle. This requires some experience.

Epoch durations usually range from 1/8 s (0.125 s) to 2 s and therefore result in a spacing between harmonics ($\Delta f$) ranging from, 8 Hz to 0.5 Hz. The first value (8 Hz based on epochs of 0.125 s duration)) is barely acceptable for sEMG because a spectrum ranging from 10 Hz to 400 Hz would be defined by only 48 harmonics. The second value (0.5 Hz, based on epochs of 2 s) would lead to 800 harmonics in the frequency range of 0 to 400 Hz. This provides a very good description of one spectrum obtained from one epoch. Nevertheless, the spectra obtained from subsequent epochs would still be different from each other and spectral averaging would be necessary (if the signal is stationary) to approximate the expected spectrum. This is indicated in Fig. 7 that shows a 4-s long recording divided into two epochs of 2 s each. Two spectra are obtained, from epoch 1 and epoch 2, with frequency resolution $\Delta f = 0.5$ Hz (harmonics are not indicated for clarity). If the two spectra are averaged, a better estimate of the desired and unknown "expected" spectrum is obtained. If many spectra, obtained from subsequent epochs, are averaged, the expected spectrum indicated by the red dashed line is progressively approximated.

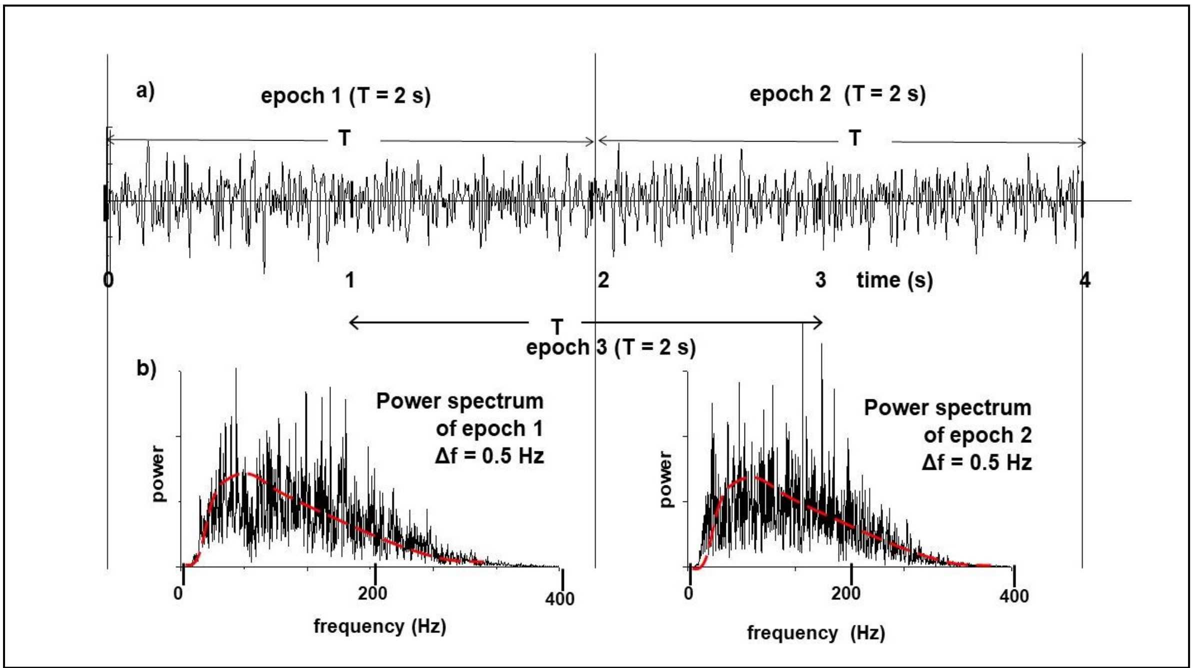


Fig. 7. Example of a 4-s recording of a sEMG signal and of "Welch periodogram". Two epochs of 2 s each are defined and two estimates of the signal's power spectrum are obtained. Spacing between harmonics (not shown) is 0.5 Hz. A third epoch with 50% overlap with epoch 1 and epoch 2 is shown. The average of the two (or three) power spectra (not shown) provides a better approximation of the expected spectra (dashed red lines in panel b). As the number N of spectra being averaged becomes large the average spectrum approaches the expected (true) spectrum of the signal. This implies the availability of long recordings of a stationary signal. This is the reason why spectral analysis of dynamic sEMG signals must be performed with caution and require expertise and competence.

To improve this approximation, the "trick" of using overlapping epochs is often used. Fig. 7 shows that a third 2-s epoch can be identified from the end of second 1 to the beginning of second 3 . Although this epoch overlaps with the others it can be used to obtain a third spectral estimate to be averaged with the other two and reduce the standard deviation of the estimates. In this case overlapping between epochs is 50% but other degrees of overlapping

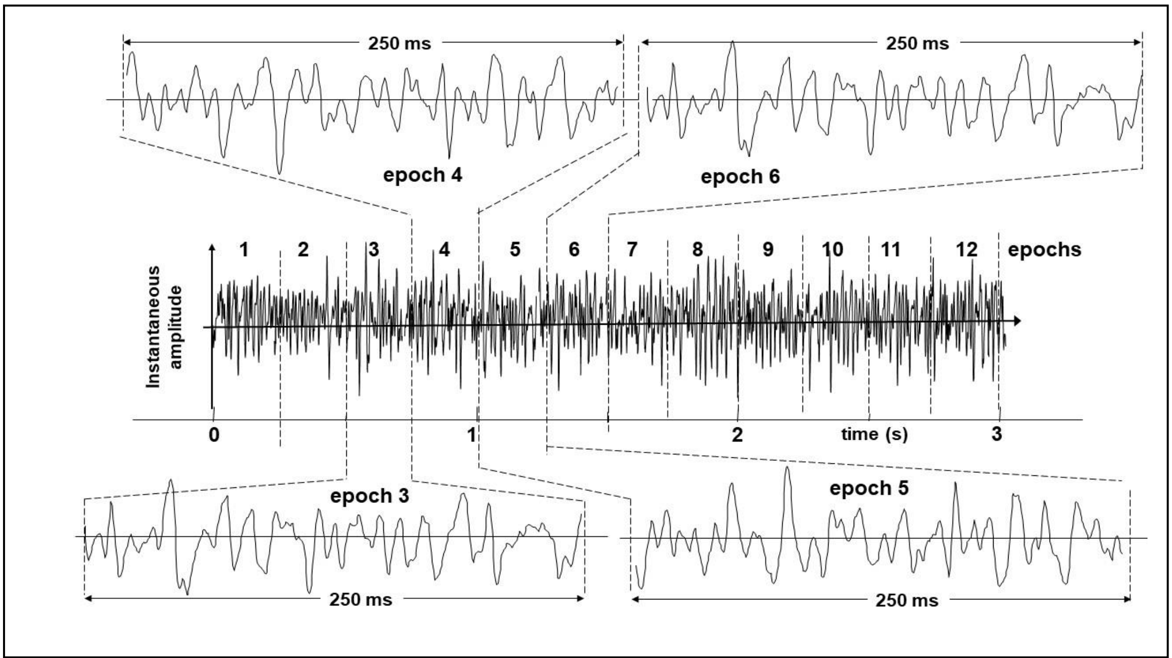


Fig. 8. Example of a 3-s recording of a random signal divided into 12 non-overlapping epochs of ¼ s each. Epochs 3 to 6 are shown. Twelve power spectra can be obtained and averaged. If epochs with 50% overlapping had been chosen, 22 spectra would have been obtained, Their average would have provided a better estimate of the power spectrum (Welch periodogram). The Welch method can be associated to the "zero padding " technique described below and in Fig. 9. It requires a stationary signal, that shows no trends in amplitude and frequency characteristics, otherwise the spectra cannot be averaged.

are often used. This procedure is called the "Welch periodogram" and is often used in sEMG analysis when the sEMG signal is stationary over at least a few seconds, such as in case of sustained isometric constant force contraction of submaximal force. Fig. 8 shows a 3-s recording of a random signal that is subdivided in 12 epochs of 0.25 s each. Only 4 non overlapping epochs are shown in the figure. The 12 epochs lead to 12 spectra that are averaged. In case of 50% overlapping 22 epochs and 22 spectra could be obtained. If the signal is stationary during these 3 s, the average of these 22 spectra is a good approximation of the expected (true) spectrum.

Another spectral estimation procedure is referred to as "zero padding". This is again a mathematical "trick" to reduce the spacing between harmonics by a particular type of interpolation. Consider Signal 1 in Fig. 9a and its spectrum defined by the blue harmonics in panel b). The Signal epoch duration is $T_1$ = 0.25 s and therefore, the blue harmonics are spaced by 4 Hz. Now consider adding null samples (padding with zero value samples) to the signal up to 0.5 s to generate an epoch of $T_2$ = 0.5 s (Signal 2, which is zero for half of its duration). The harmonics of Signal 2 are now spaced by 2 Hz (red and blue harmonics in panel b). Apparently we have increased the frequency resolution of the spectrum but this is not really the case since the information content of the two spectra is the same and we have just performed a special type of interpolation in the frequency domain. Nevertheless, this is a useful procedure adopted when epochs are short either because only a short signal recording is available or because the signal is not stationary and short epochs must be used. See also section 5.

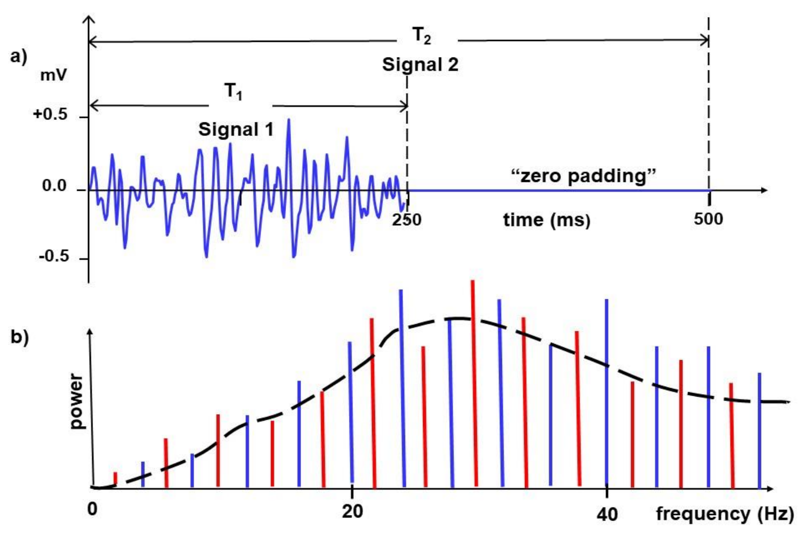


Fig. 9. Example of the "zero padding technique" to obtain a higher number of harmonics through a special interpolation technique in the frequency domain. This method can be used when short signal epochs are extracted from a non stationary signal, such as a sEMG signal during dynamic contractions. This approach does not add new information but allows a better estimate of spectral features because of the greater number of harmonics more closely spaced. The workings of this technique are not intuitive but are fully mathematically justified.

It is important to underline that the harmonics of a signal are mathematical objects providing a different representation of a signal that is often very useful for the interpretation of the signal nature and its changes. Harmonics are not produced by physiological processes. They provide a different way of describing and quantifying such processes, like a description in a different language.

## 5. The effect of sampling frequency and duration of a signal on its Fourier transform.

The Nyquist-Shannon theorem states that a sinusoid can be reconstructed if **more** than two samples are taken for each period. For further information see: https://en.wikipedia.org/wiki/Nyquist%E2%80%93Shannon_sampling_theorem and https://www.youtube.com/watch?v=FcXZ28BX-xE

This does not mean that the reconstruction is trivial. It actually requires sophisticated procedures much more complex than linear interpolation. Fig. 10 shows the results of sampling, at 100 samples/s, two sinusoids at 22 Hz (4.54 samples/period) and at 27 Hz (3.70 samples/period). A 25 Hz sinusoid sampled at 100 samples/s (4 samples per period) would result in a triangular wave but could still be reconstructed as a sinewave.

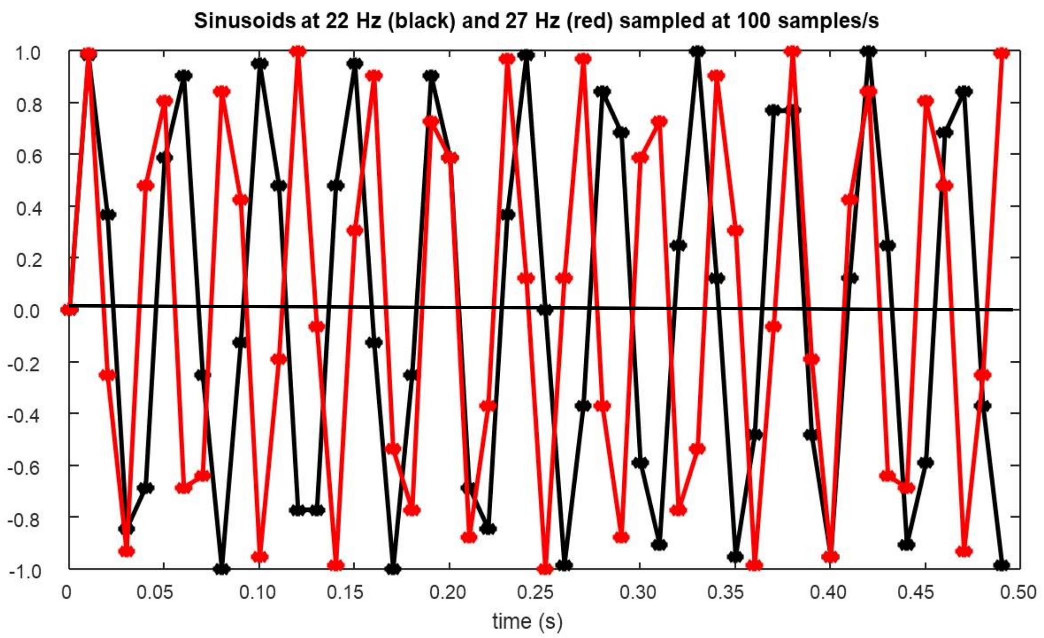


Fig. 10. Two sinusoids of peak values equal to 1 ( at 22 nd 27 Hz) are sampled at 100 Hz. The samples are connected by interpolating straight lines. Despite the fact that the Nyquist-Shannon theorem is satisfied, the original signals are poorly approximated but could be recovered with proper mathematical tools. Since these tools are complex, a signal is usually sampled at 5-10 times the prescribed Nyquist rate, that is at 5-10 times the frequency of its highest harmonic.

The highest harmonic of the sEMG signal is around 400 Hz (occasionally up to 500 Hz) as indicated in Fig. 7. The sEMG should therefore be sampled at more than 800-1000 samples/s in order to reconstruct (with the proper technique) all its harmonics and, therefore, the entire signal without altering it. Higher "harmonics" (due to noise or other disturbing signals) above 400-500 Hz should be eliminated, before sampling the signal, by analog filtering. Since the contribution of the harmonics of the sEMG in the range of 350-500 Hz is small (Fig. 7) a compromise sampling frequency near 2000 Hz is usually adopted. A sampling frequency of 2048 samples/s is commonly adopted because the algorithm for the computation of the Fourier transform is much faster if the sampling frequency is a power of 2 ($2^n$) and $2048=2^{11}$. With this choice the commonly used epochs of 0.250s, 0.50 s, 1.00 s, 2.00 s, have a number of samples that are powers of 2 (512, 1024, 2048, 4096 respectively). However, other sampling frequencies near or above 2000 samples/s are acceptable and used.

As indicated in Fig. 4 and in previous sections, the period of the first harmonic of a random signal equals the duration T of the signal epoch selected for the computation of the Fourier transform. Therefore, the frequency of the first harmonic is 1/T Hz and the spacing between harmonics is also 1/T Hz where T is the chosen epoch duration. As a consequence, the greater the epoch duration the greater the number of harmonics describing the signal and the better the estimate of the amplitude and power spectra of the signal. Note that the often used "zero padding" technique described above to increase the number of harmonics provides only an interpolation and does not provide additional information.

Since the sEMG has harmonics up to about 400 Hz (occasionally up to 450-500 Hz) it is usually a good idea to have at least 100 harmonics which means a spacing between harmonics of <4 Hz and an epoch duration of at least 0.25 s. This may be too long in fast dynamic contractions. Averaging of spectra is not possible in this case, because the signal and its spectrum are changing too much and too fast in time and a compromise may be necessary. Other techniques (such as "time-frequency analysis" or "wavelet analysis") may solve this problem but they are complex and are not discussed in this note. Other issues related to sampling a signal in space are addressed in similar ways but are not discussed in this note.

If the signal has a DC component (that is the mean value not zero) this component appears as an "harmonic of order 0" for f=0 Hz in the amplitude or power spectra Since, in general, this value is of no interest and may alter estimates of frequency parameters (such as the mean frequency), the mean value of the signal is subtracted from the signal before any further analysis is performed. In this way the $0^{th}$ harmonic at f=0 has zero amplitude.

## 6. Application examples. Applications to the sEMG signal.

Fig. 11 shows a signal made up of three sinusoids (with respective frequencies of 80 Hz, 160 Hz and 200 Hz) and a random signal. These four contributions have similar peak-to peak amplitudes (panels a), b), c), d) ). Their sum produces the signal depicted in panel e).
It is virtually impossible to visually "understand" the nature and structure of the signal by looking at it in panel e). However, the power spectrum of the sum signal (in panel e) ) clearly shows that the signal is composed of three sinusoids plus noise, it indicates the frequencies of the sinusoids and the presence of noise (panel f) ). The noise has many small harmonics (not indicated) distributed on a wide frequency range.

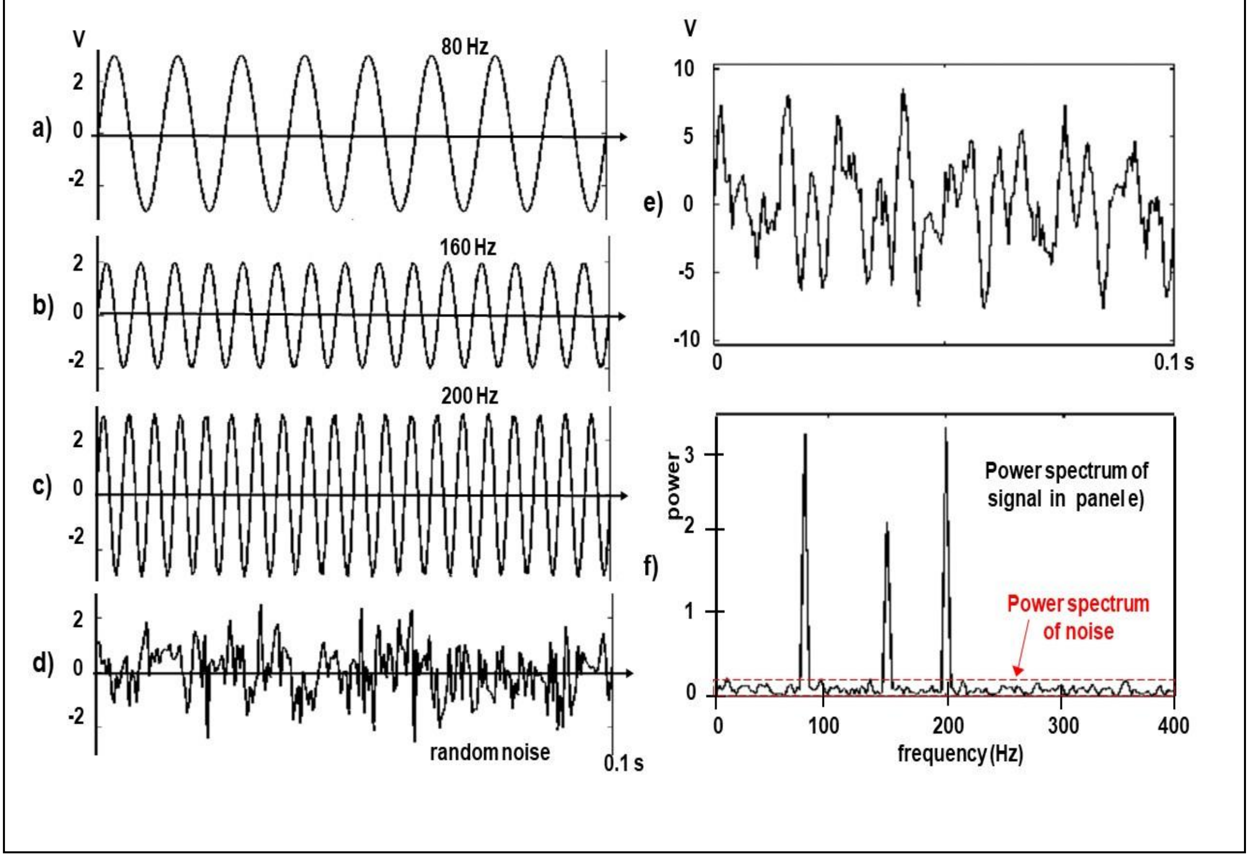


Fig. 11. The summation of the sinusoids and noise indicated in a), b) , c) and d) produce the signal depicted in e) where the original components can no longer be visually recognized. However, the power spectrum of the signal e), depicted in f) , clearly shows the three sinusoids and the noise that make up the signal in e).

Fig. 12 shows the power spectrum of a sEMG signal. The black plot depicts the power spectrum obtained by squaring the amplitude of the Fourier transform obtained from a single

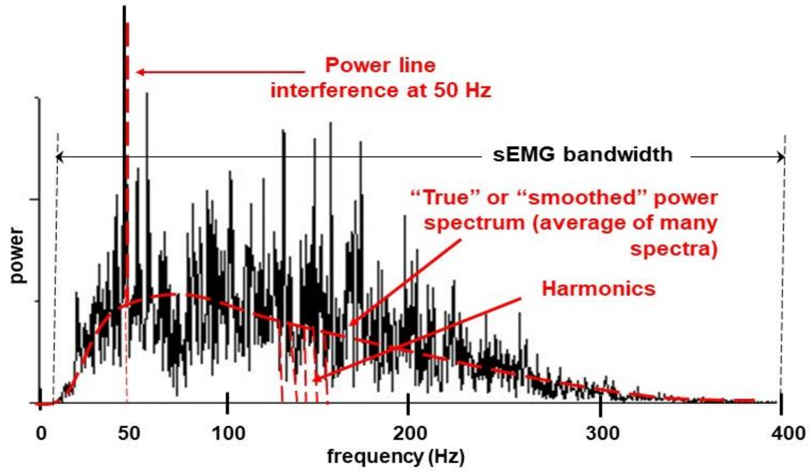


Fig. 12. Black line: power spectrum of one epoch of a random-like signal (sEMG) containing a marked spectral line at 50 Hz likely due to power line interference. Only a few harmonics are indicated for clarity. Red line: power spectrum resulting from the averaging of many power spectra, each computed on a single epoch, and approximating the true spectrum of the signal. A few harmonics of the averaged spectrum are indicated in red. The spectral line at 50 Hz, present in all the averaged spectra, is very evident. Many methods are available to reduce or eliminate such interference., when detected.

epoch. Since the signal is random-like also the amplitudes of the harmonics are random and many spectra, obtained from many signal epochs of a stationary signal, must be averaged to obtain an average spectrum providing a good estimate of the true spectrum indicated in red. This spectrum shows a very large harmonic at 50 Hz presumably due to power line interference and not visually detectable by looking at the signal in time (not shown). This "spectral line" can be removed by a "notch filter" or by other techniques (not described in this note) to obtain a signal free from this interference.

The two examples given in Fig. 11 and Fig. 12 show the power of the Fourier analysis in signal processing and "understanding". A third example specifically focuses on a technique frequently used to provide an index of "myoelectric manifestations of muscle fatigue".
As a muscle "fatigues" the propagation velocity of action potentials along its fibers decreases making the motor unit action potentials (MUAP) wider. The sum of the MUAPs is the sEMG detected by the surface electrodes. The global effect on the sEMG is a progressive "slowing" of the signal and therefore a "narrowing" or "compression" or "scaling" of its

amplitude and power spectra, as indicated in Fig. 6 and Fig. 13. Fig. 13 shows a long record of one sEMG channel subdivided into N epochs of equal durations, the power spectra of epochs 1 (red) and N (blue), and the mean spectral frequencies (MNF or centroid lines) of the first and last spectra.

Since each epoch is 0.5 s long, the spectral harmonics (not indicated for clarity) will be separated by 2 Hz in all spectra. Their amplitudes will change (not their frequencies) showing increased power of the lower harmonics and decreased power of the higher harmonics, leading to a decrease of MNF. Since the sEMG is slowly changing its properties in time is said to be “quasi-stationary” and is assumed to be stationary during each epoch.

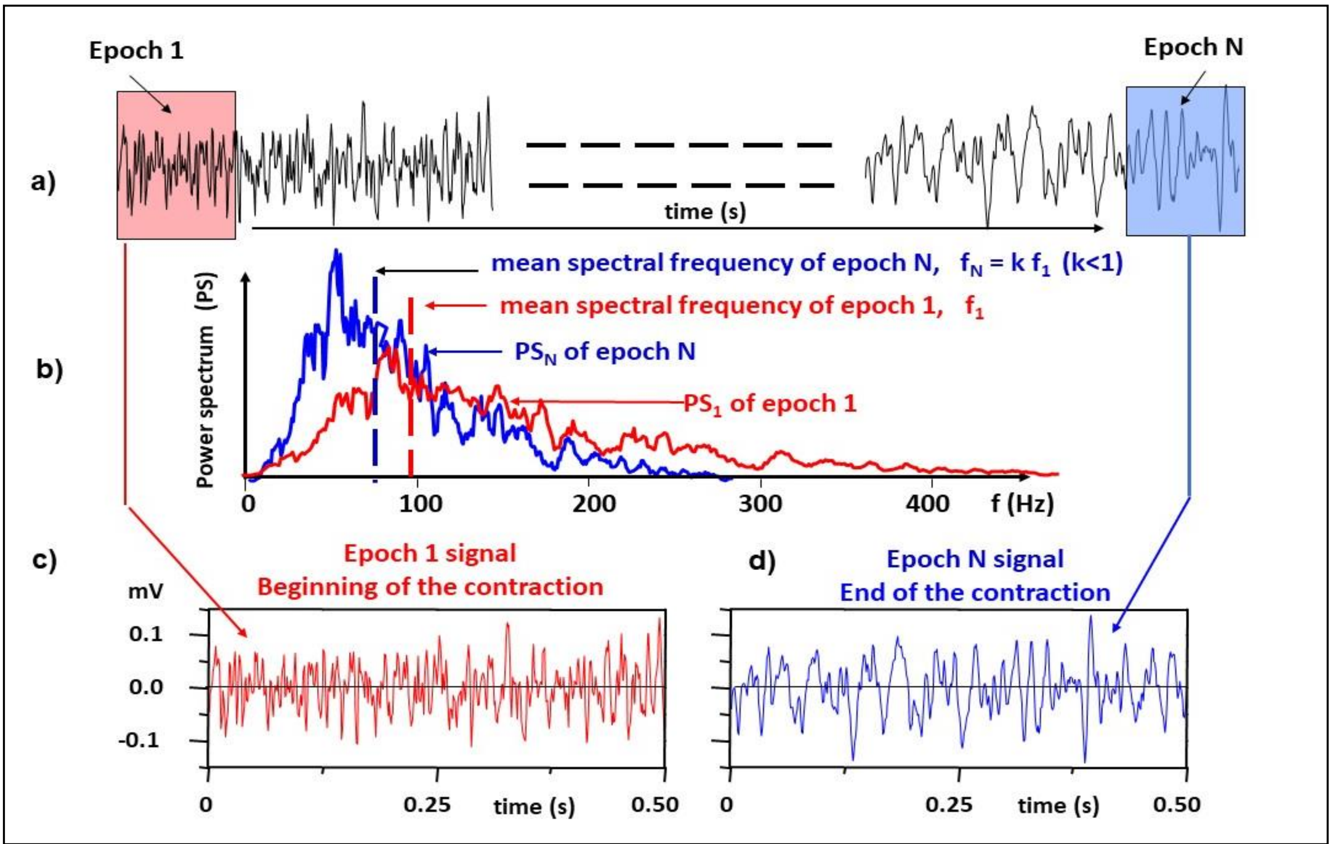


Fig. 13. A long recording (e.g. 60 s) of non stationary sEMG is divided into N epochs of a few seconds each and the power spectrum is calculated for each epoch. For clarity, only epoch 1 and N are shown with the relative power spectra $PS_1$ and $PS_N$. The mean frequency (MNF) $f_1$ to $f_N$ of each spectrum is calculated. If, for example , $f_N$ is 0.7 $f_1$ (that is 70% of $f_1$) this means that MNF decreased by 30% during the contraction. This decrement can be used as a quantitative index of myoelectric manifestation of muscle fatigue.

The decrease of MNF during a sustained isometric, constant force contraction may be somewhat irregular from epoch to epoch leading to the situation depicted in Fig. 14 where a regression line is fitted to the experimental points (one per epoch) describing the variations of MNF versus time. The slope of this regression line is often taken as an index of myoelectric manifestations of muscle fatigue.

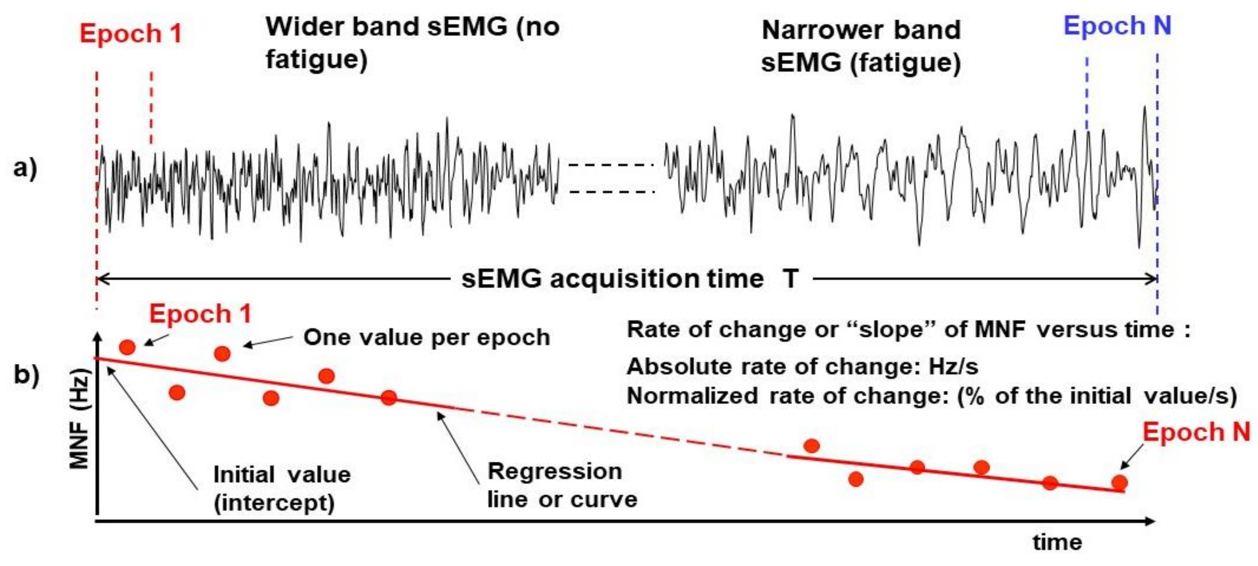


Fig. 14. Myoelectric manifestations of muscle fatigue. a) A "quasi-stationary" (with slowly changing features) sEMG signal is recorded during a constant force isometric contraction for a total time T. This long interval is divided into N epochs (epoch 1 to epoch N) of equal duration $T_{epoch}$.
b) The power spectrum of the signal is estimated for each of these epochs and its mean frequency (MNF) is calculated and plotted versus time (red dots). A curve fitting is performed and the slope of the curve (a straight line in this case) is the rate of change of MNF in time. This slope, expressed in Hz/s or in % decrement with respect to the initial value, is taken as an index of myoelectric manifestation of muscle fatigue.

Fig. 15 shows the same concept with a three-dimensional plot showing the evolution of the sEMG power spectrum during an isometric constant force contraction of a biceps brachii sustained for 90 s at 60% of its maximal voluntary contraction (MVC). More information about myoelectric manifestations of muscle fatigue and about the "fatigue plot" showing the time course of other sEMG features can be found in module 7 at https://www.robertomerletti.it/en/emg/material/teaching/.
Caution and competence must be used in the interpretation of these plots since the concept of myoelectric manifestations of muscle fatigue is very different from the concept of mechanical manifestations of muscle fatigue. Myoelectric manifestations of muscle fatigue are due to electrophysiological changes taking place in the sarcolemma. They are evident even when

contraction force remains constant and begin at the beginning of the contraction, when mechanical manifestations of muscle fatigue are not yet present (the muscle is fatiguing but still able to sustain the required force). Other sEMG variables reflect fatigue. See the concept of "fatigue plot" in https://www.robertomerletti.it/en/emg/material/teaching/module7.

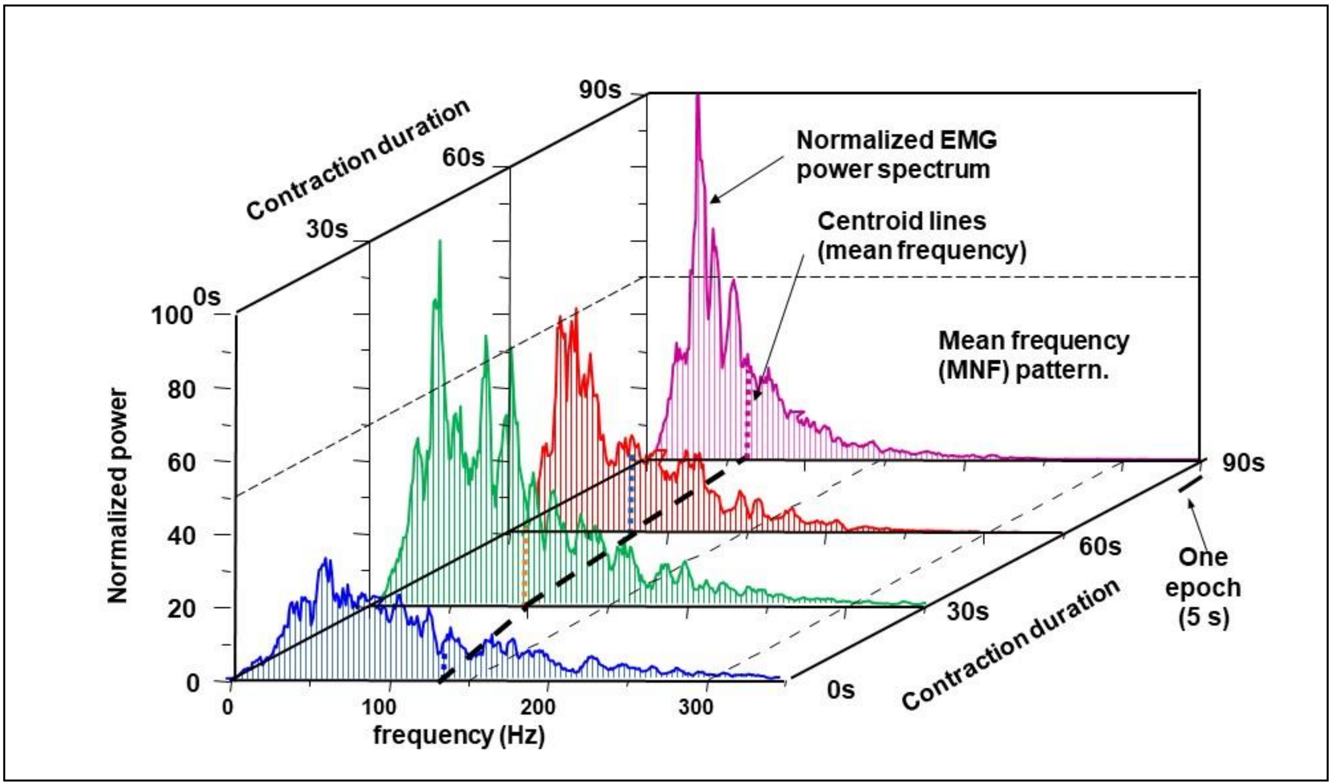


Fig. 15. Example of power spectrum of the sEMG of a biceps brachii during a sustained isometric constant force contraction sustained at 60% of the maximal voluntary contraction (MVC) for 90s. Power spectra are calculated over subsequent epochs of 5 s each. For clarity, only four spectra are depicted, at the beginning of the contraction, after 30 s, after 60 s and at the end of the contraction. The vertical hatching of each spectrum represents the harmonics which are at the same frequencies for all the spectra since the epochs have equal durations (5s). The centroid value of each spectrum (MNF, vertical thick dotted line) progressively moves towards the lower frequency values demonstrating myoelectric manifestations of muscle fatigue (dashed black curve). The rate of change can be taken as an index of fatigue and can be expressed in Hz/s or in % decrement/s with respect to the initial value.

## 7. The cross-spectrum of two signals.

The power spectrum (or "auto power", $P_{xx}$) of a signal x(t) indicates the power contributed to the signal by each harmonic. If the signal is random the power spectrum is somewhat different from epoch to epoch and shows random fluctuations from epoch to epoch so that each harmonic has a mean value and a standard deviation. An estimate of the "true" spectrum is obtained by averaging many spectra obtained from many epochs, as indicated above. The power spectrum is the square of the magnitude of the Fourier transform of the signal, that is $|P_{xx}(f)| = |X(f)|^2 = |X(f)| \cdot |X(f)|$, where X(f) is the Fourier transform of x(t) and | | means magnitude .

Consider now two stationary signals x(t) and y(t) having Fourier transforms X(f) and Y(f) computed over the same epoch duration T and therefore having harmonics at the same frequencies 0 Hz, 1/T, Hz 2/T Hz, 3/T Hz etc. We may be interested in finding out if these signals have something in common. In this case their power spectra, computed on the same sequence of epochs, would have the amplitude of some of their harmonics fluctuating in a similar way, that is in a "correlated manner" going up and down together. This may mean that the two signals are (at least in part) due to a common source or that one in influencing the other. This fact may be of great interest in neurophysiology and is described in a quantitative manner by the "cross-spectral power" or " cross-spectrum" of the two signals. The magnitude of this cross-spectrum is given by $|Pxy(f)| = |X(f)| \cdot |Y(f)|$ and shows the harmonics of the two signals that carry "coherent" information. A version of $|Pxy(f)|$ "normalized" with respect to the power spectra of the two signals is $|Pxy(f)|^2 / (P_{xx}(f) \cdot P_{xx}(f)$ and is called the "coherence function" between the two signals. This function is comprised between 0 and 1 (or 0 % and 100 %) and quantitatively indicates the "commonality" between two signals.

Further discussion about these issues exceeds the purpose of this note but a more detailed explanation may be found in:

https://psyarxiv.com/mj75a/

https://math.stackexchange.com/questions/1002/fourier-transform-for-dummies

https://www.medizin.uni-muenster.de/fileadmin/einrichtung/sfbtrr58/downloads/PhD_Students/mathlab-for-neuroscientists.pdf

https://en.wikipedia.org/wiki/Nyquist%E2%80%93Shannon_sampling_theorem and

https://www.youtube.com/watch?v=FcXZ28BX-xE

Additional material can be found in the book :

Afshin Samani, An introduction to signal processing for non-engineers. 2020, CRC Press, Taylor an Francic Group (CRC Press), ISBN: 13:978-0-367-20755-7